THE HAUPTVERMUTUNG FOR 3-COMPLEXES

BY
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1. **Introduction.** In two fundamental papers, [13] and [14], Poincaré introduced the notions of a complex, and of a triangulation. The attempt to study certain topological spaces by means of a triangulation runs into an obvious problem, one must show that invariants defined in terms of the triangulation are independent of the choice of triangulation. For most invariants it is relatively easy to show that they remain unchanged when the triangulation is subdivided, but when an entirely new triangulation is substituted the situation is far from clear.

Thus, for example, Tietze in [17] concerns himself with, among other things, showing that the Betti numbers, the torsion coefficients, the fundamental group, etc., are topological invariants.

In [16] Steinitz takes, possibly for the first time, the point of view that complexes will be treated in their own right and that the notion of isomorphism will be combinatorial equivalence. This seems to be the reason that Steinitz gets the blame for the Hauptvermutung.

Cases of the problem were treated by several people after that, an excellent account is given in [4]. Cairns was probably not aware when he wrote his article that Papakyriakopoulos in [12] had proved the Hauptvermutung for two dimensional complexes. In a series of papers E. Moise proved both the triangulation theorem and the Hauptvermutung for 3-manifolds without boundary in [9], and for bounded 3-manifolds in [10], R. H. Bing in [1] also extended the results of [9] to the case of a bounded 3-manifold.

In [5], Gluck gave an elegant proof of a weak version of the Hauptvermutung for triangulated manifolds homeomorphic to cells or spheres. Smale proved the Hauptvermutung for combinatorial n-manifolds which are homeomorphic to cells or spheres, $n \neq 4, 5, 7$ in [15].

The question took an unexpected turn when Milnor in [8] exhibited, for each $n \geqq 6$, a pair of compact n-dimensional complexes which, while homeomorphic as spaces, are combinatorially distinct.

The author in his thesis [2] proved that a homeomorphism between 3-complexes could be approximated by a piecewise-linear homeomorphism. This result was also announced by Ross Finney.

In the present paper we publish not only that result but we also show that a homeomorphism between 3-complexes may be moved by an isotopy to an

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approximating piecewise-linear homeomorphism. In [7] Kister showed that two sufficiently close homeomorphisms of a 3-manifold to itself are isotopic. Putting this together with the results of Bing and Moise [1], [10], one concludes our theorem for the case of 3-manifolds with boundary. We shall show however, in addition to the above, that the isotopy need not change the homeomorphism where it was already piece- wise linear.

Since the proof is very long we shall give the thread of it here. The basic idea is to reduce the theorem to the cases for a 3-manifold and for the cone over a 2-manifold. The tool we use for this reduction was introduced by Munkres in [11] to study locally polyhedral spaces. We shall use a simplified version for complexes. If we call those points which have no open cell neighborhood the singular points, then the tool is a construction which tears apart the complex along the set of singular points. In three dimensions the result is a complex, called the composition space, such that the star of each vertex is the cone over a connected 2-manifold.

§§2 and 3 are devoted to the general theory of these composition spaces, and in particular in §3 the necessary approximation results are proved. In §4 we come to the proof of the Hauptvermutung itself. The two dimensional result is proved in the form we need. The major difficulty to the proof in three dimensions is what happens on the singular set. In forming the composition space each nonsingular point is replaced by a single nonsingular point. Each singular point however is replaced by several points in general. When one changes the homeomorphism to make it piecewise-linear on the composition space one must be very careful about what happens on the singular set since points which will be matched back together must be carried to matching points again.

To take care of this difficulty we proceed as follows. We first make our homeomorphism piecewise-linear on a certain 1-dimensional subcomplex. Then restricting it to the singular set (which is 2-dimensional) we use the Hauptvermutung for two-complexes to move it isotopically to a piecewise-linear homeomorphism, not changing it on the 1-complex. Having chosen the 1-complex appropriately, and having not moved things too far, we can lift the homeomorphism and the isotopy to the composition space. We then extend the isotopy to the composition space, and then compose with a further isotopy which makes it piecewise-linear, but does not change it on the singular set. This new isotopy fits back together to give an isotopy on the original complex.

A word about general notation. We use *complex* to mean locally finite, finite dimensional, simplicial complex, and use symbols like $K, L, \dots$ for them. An element of a complex is an *open* simplex, denoted $\sigma, \tau, \dots$. If σ is a face of τ we write $\sigma < \tau$. We shall usually find it necessary to distinguish between a complex K, and its underlying topological space, or polyhedron, $|K|$. If A is a subset of the polyhedron $|K|$, we denote the *open star of A in K* by st $(A; K)$, or just st (A) if K is understood. It is the set of (open) simplexes of K some face of which meets A. The same symbol will be used for the point set which is the union of those simplexes.

It is an open neighborhood of A in $|K|$. We denote by St $(A; K)$ the set of all simplexes which are faces of simplexes in st $(A; K)$, and also the polyhedron of this complex. As a point set St $(A; K)$ is closed in $|K|$. If σ and τ are in K and have no face in common (i.e. $\bar{\sigma} \cap \bar{\tau} = \varnothing$), but are both faces of a single simplex, we say they are *joinable*, and we denote by $\sigma * \tau$ the smallest simplex having both as face. We call $\sigma * \tau$ the *join* of σ and τ. Again if $\sigma \in K$ we denote by lk $(\sigma; K)$, or just lk (σ), the set of all simplexes of K which are joinable to τ. We call lk $(\sigma; K)$ the *link* of σ in K and we shall not distinguish this complex from its polyhedron.

A few words about maps and isotopies. A PL *map* $f: |K| \to |L|$ is a continuous function so that K and L have subdivisions for which f is simplicial. To be simplicial f must not only map simplexes to simplexes, but must also preserve the linear structure within the simplex (i.e. be linear in the barycentric coordinates). If f and g are homeomorphisms from $|K|$ to $|L|$ we say f is isotopic to g when there exists a map $F: |K| \times [0, 1] \to |L|$ so that if $f_t(x) = F(x, t)$ then $f_0 = f$, $f_1 = g$, and f_t is a homeomorphism for each t. We call the family f_t an isotopy. We shall follow Zeeman [19] in saying such ugly but convenient things as "Isotope f to g" when we mean "Choose an isotopy f_t with $f_0 = f$ and $f_1 = g$", etc.

The author would like to take this opportunity to thank his thesis advisor James Munkres, and Hermann Gluck for the help and encouragement they gave him.

2. **The composition space.** In [11] Munkres defined what he called the composition space of a locally polyhedral space. We shall make use of this notion to turn a 3-complex into something approaching a manifold. Since we are starting with a complex we can considerably simplify Munkres definitions.

(2.1) *Let K be a complex and σ a simplex of K. If x and y are any points of σ then there is a homeomorphism f of $|K|$ to itself carrying x to y.*

Since $\bar{\sigma}$ is a ball, there is a homomorphism f_1 of $\bar{\sigma}$ to itself which carries x to y, and which is identity on the boundary of the ball. Define f to be the identity on $|K| - \text{St}(\sigma)$. If $\tau \in \text{lk}(\sigma)$ define f on $\sigma * \tau$ to be the linear extension of f_1 on σ with the identity map on τ.

(2.2) *If $A \subset |K|$ is carried onto itself by every homeomorphism of $|K|$ to itself, then A contains every simplex of K which it intersects. If, in addition, A is closed then A is the polyhedron of a subcomplex of K.*

If K is any complex, and x is a point of $|K|$, we shall denote by $d(x)$ the *local dimension* of $|K|$ at x (see [6]). Notice that $d(x) = n$ provided x lies on the closure of an n-simplex of K but not on the closure of an $(n+1)$-simplex. We denote by $i(x)$, called the *index* of x, the largest integer m so that x is a point of an m-simplex in a triangulation of some neighborhood of x. This is what Munkres [11] calls the index of x in the trivial locally polyhedral structure.

The functions d and i are topological invariants, so we may apply (2.2) to prove:

(2.3) *The functions d and i are constant over simplexes of K.*

Let us use $b|K|$ to denote the set of points x of $|K|$ for which $i(x) < d(x)$. We call $b|K|$ the set of *singular points* of $|K|$. It consists of exactly those points of $|K|$ which have no neighborhood homeomorphic to the interior of a cell. We might say that $b|K|$ is the set of points where $|K|$ fails to be a manifold. In particular the components of $|K| - b|K|$ are manifolds.

Clearly $b|K|$ and $|K| - b|K|$ are carried onto themselves by any homeomorphism of $|K|$ to itself. If we show that $b|K|$ is closed in $|K|$ we can then apply (2.2) to show:

(2.4) *The set $b|K|$ is the polyhedron of a subcomplex bK of K.*

Let x be a point of $|K| - b|K|$, and let U be a triangulated neighborhood of x so that x lies on a simplex σ of U having dimension $i(x)$. Since $i(x) = d(x)$, σ is not the face of any simplex of U. Then σ is a neighborhood of x in U, and hence in $|K|$. Since d and i are constant over σ, $\sigma \subset |K| - b|K|$.

If K is any complex we shall denote by K' a first derived complex of K (see [19]). A first derived complex of K differs from the first barycentric subdivision only in that the new vertex added to the interior of each simplex is not necessarily the barycenter. If σ is any simplex of K we shall denote by σ' the vertex of K' interior to σ.

We come now to the lemmas which are at the heart of the relation between the local topological structure and the simplicial structure of the complex.

(2.5) *Let σ be any simplex of the complex K and let τ be a simplex of K' contained in σ, and of the same dimension as σ. Then each component of* $\mathrm{st}\,(\sigma; K) - b|K|$ *contains exactly one component of* $\mathrm{st}\,(\sigma'; K') - b|K|$, *and exactly one component of* $\mathrm{st}\,(\tau; K') - b|K'|$. *In particular each component of* $\mathrm{st}\,(\sigma'; K') - b|K'|$ *contains exactly one component of* $\mathrm{st}\,(\tau; K') - b|K'|$.

We begin with the proof of the first assertion. If p is any point on the boundary of $\mathrm{st}\,(\sigma; K)$ then the ray $[\sigma' p]$, from σ' to p, meets the boundary of $\mathrm{st}\,(\sigma'; K')$ in a single point q. We map $[\sigma' p]$ linearly onto $[\sigma' q]$ leaving σ' fixed. This gives a (non-piecewise-linear) homeomorphism of $\mathrm{st}\,(\sigma; K)$ onto $\mathrm{st}\,(\sigma'; K')$ which maps simplexes of $\mathrm{st}\,(\sigma; K)$ into themselves. Since bK is a subcomplex of K, this map carries $\mathrm{st}\,(\sigma; K) - b|K|$ homeomorphically onto $\mathrm{st}\,(\sigma'; K') - b|K'|$ and carries components of $\mathrm{st}\,(\sigma; K) - b|K|$ into themselves.

The second assertion is proved in the same manner, except that we consider rays from the barycenter τ' of τ rather than from σ'.

If x is a point of $|K|$ Munkres [11] defines the *singularity* of x, denoted $s(x)$, as the smallest integer m so that for arbitrarily small neighborhoods U of x in $|K|$, $U - b|K|$ has m components. Since $s(x)$ is defined in a topologically invariant fashion, the function $s(x)$ is constant on simplexes of K by (2.2). The next lemma shows how the value of $s(x)$ can be determined from the combinatorial structure of K.

(2.6) *If σ is a simplex of K then the constant value of $s(x)$ on σ is equal to the number of components in* $\mathrm{st}\,(\sigma; K) - b|K|$.

If U is a neighborhood of a point x of σ with $U \subset \mathrm{st}(\sigma; K)$, then $U - b|K|$ has at least as many components as $\mathrm{st}(\sigma; K) - b|K|$. On the other hand there are arbitrarily small open sets U about x so that $U - |b(K)|$ has the same number of components as $\mathrm{st}(\sigma; K) - |b(K)|$. Indeed one may choose $U = \mathrm{st}(x; K^{(r)})$ where $K^{(r)}$ is an rth derived complex of K, and x is a vertex of $K^{(r)}$ but not of $K^{(r-1)}$. Then (2.5) applied r times proves the lemma.

We come now to the definition of the composition space of Munkres. We have chosen a somewhat different route to the definition than that used by Munkres for two reasons. We shall need the particular triangulation that we introduce. But also, since we do not need anything like Munkres generality, we can give a somewhat simpler definition. Indeed we are defining what Munkres calls the composition space in the trivial locally polyhedral structure on $|K|$.

Let K be any complex. The *composition space* of K is the pair $(\tilde{K}, p)$ consisting of the complex $\tilde{K}$ and the simplicial map $p: \tilde{K} \to K'$, of $\tilde{K}$ onto K' defined as follows:

Let σ be a simplex of K'. For each component $C(\sigma, i)$ of $\mathrm{st}(\sigma; K') - b|K'|$ we associate a simplex σ_i of $\tilde{K}$ having the same dimension as σ. We define $p(\sigma_i) = \sigma$, and $\tau_j < \sigma_i$ provided $p(\tau_j) < p(\sigma_i)$ and $C(\tau, j) \supset C(\sigma, i)$. It is easy to see that if σ_i in $\tilde{K}$ has dimension m, then σ_i has $m+1$ vertices, that any subset of these is the set of vertices of a face of σ_i, and that the images of these vertices under p determine $p(\sigma_i)$. The only thing left to show is that the vertices of σ_i determine σ_i, that is that no two simplexes of $\tilde{K}$ have the same set of vertices.

If another simplex of $\tilde{K}$ has the same vertices as σ_i then its image under p is also σ, so it is σ_j for some j. By (2.5) there is a vertex v of σ so that each component of $\mathrm{st}(v; K') - b|K'|$ contains exactly one component of $\mathrm{st}(\sigma; K') - b|K'|$. Thus σ_i and σ_j have different elements of $p^{-1}(v)$ as vertex. Of course it is clear from the definition that p is a simplicial map preserving dimension of simplexes.

It would be more satisfying if p were to map $\tilde{K}$ simplicially onto K rather than K'. But if one tries to define $\tilde{K}$ using simplexes of K rather than those of K', then it sometimes happens that two simplexes of $\tilde{K}$ have the same set of vertices. This happens if K is the triangulated plane together with an extra 2-simplex having one edge in common with the plane.

It is reasonably clear that $(|\tilde{K}|, p)$ is a topological invariant of $|K|$, and one could refer to Munkres [11] for a proof of this. What is perhaps less obvious is that $(|\tilde{K}|, p)$ is functorial, at least as far as homeomorphisms are concerned. The next lemma and theorem prove this fact, and incidentally also prove the topological invariance.

(2.7) *Let $(\tilde{K}, p)$ be the composition space of K. Then*

(a) *p maps $|\tilde{K}| - p^{-1}(b|K|)$ homeomorphically onto $|K| - b|K|$.*

(b) *$|\tilde{K}| - p^{-1}(b|K|)$ is dense in $|\tilde{K}|$.*

(c) *If $\{x_n\}$ is a sequence in $|\tilde{K}| - p^{-1}(b|K|)$ converging to a point of σ_i, then for n large, $p(x_n) \in C(\sigma, i)$.*

We begin with a proof of part (a). If σ is a simplex of $K'-b(K')$, then st $(\sigma; K') \subset K'-b(K')$ since $b(K')$ is a subcomplex of K'. Thus st $(\sigma; K')-b|K'|$ is connected, and p maps st $(\sigma_1; \tilde{K})$ isomorphically onto st $(\sigma; K')$.

To show (b) and (c) we let τ_i be any simplex of $p^{-1}(b(K'))$, and let σ be any simplex of K' contained in $C(\tau, i)$. Then $p^{-1}(\sigma)=\sigma_1$ is a single simplex of $\tilde{K}-p^{-1}(b(K'))$ having τ_i as a face. Both (b) and (c) follow.

(2.8) (Isotopy lifting theorem). *Let K and L be complexes, and $f_t: |K| \to |L|$, $0 \leq t \leq 1$ be an isotopy. For each t there exists a unique continuous map $\tilde{f}_t: |\tilde{K}| \to |\tilde{L}|$ so that the diagram*

$$\begin{tikzcd} |\tilde{K}| \arrow[r, "\tilde{f}_t"] \arrow[d, "p"'] & |\tilde{L}| \arrow[d, "p"] \\ |K| \arrow[r, "f_t"] & |L| \end{tikzcd}$$

commutes. The map $\tilde{f}_t$ is a homeomorphism, and if f_t is piecewise linear on the subcomplex M of K then $\tilde{f}_t$ is piecewise linear on $p^{-1}(M')$. Finally the family $\tilde{f}_t$ is an isotopy.

Notice that if f_t is a constant isotopy, we are lifting a homeomorphism.

Since f_t is a homeomorphism, $f_t(|K|-b|K|)=|L|-b|L|$. Thus we may define $\tilde{f}_t \mid |\tilde{K}|-p^{-1}(b|K|)=p^{-1}\cdot f_t\cdot p$, and we must make this definition if we want the diagram to commute. Since $|\tilde{K}|-p^{-1}(b|K|)$ is dense in $(\tilde{K})$ by (2.7b), there is at most one continuous extension of $\tilde{f}_t$ to all of $|\tilde{K}|$. This proves the uniqueness.

Let $y \in b|K|$ and let $y_1, \ldots, y_s$ be the points of $p^{-1}(y)$ corresponding to the components $C(y, 1), \ldots, C(y, s)$ of st $(y; K')-b|K|$. Choose $r \geqq 1$ so large that $f_t(\text{Cl}\,[(\text{st}\,(y; K^{(r)})]) \subset \text{st}\,(f_t(y); L')$. By (2.5) the sets $D(y, i)=C(y, i) \cap \text{st}\,(y; K^{(r)})$ are the components of st $(y; K^{(r)})-b|K|$. Since $f_t(b|K|)=b|L|$ there is a unique integer $j(i)$ so that $f_t(D(y, i)) \subset C(f_t(y), j(i))$. We define $\tilde{f}_t(y_i)=x_{j(i)}$ where $x=f_t(y)$. This extends $\tilde{f}_t$ to a function on all of $|\tilde{K}|$.

We claim that the function $\tilde{f}_t(x)$ so defined is simultaneously continuous in both x and t. To see this we shall use a theorem from point set topology which says that if a function is continuous when restricted to each member of a locally finite closed cover of a space, then the function is continuous. The space we have in mind is $|\tilde{K}| \times I$, $I=[0, 1]$, and the cover is $\{\bar{\sigma}_1 \times I\}$ where σ_1 ranges over the simplexes of $\tilde{K}-p^{-1}(b(K'))$.

Keeping the notation introduced in the preceding paragraph but one, we let (x_n, t_n) be a sequence in $\sigma_1 \times I$ converging to the point (y_i, t) of $p^{-1}(b|K|) \times I$. By (2.7c) the sequence $z_n=p(x_n)$ is contained in $D(y, i)$ if we discard a few initial terms. If n is large enough we will also have $f_{t_n}(D(y, i)) \subset C(f_{t_n}(y), j(i))$. Now $x_{j(i)}$ is the only point of $p^{-1}(x)$ in the closure of $p^{-1}(C(x, j(i)))$ (recall $x=f_t(y)$) for otherwise two components of st $(x; L')-b|L'|$ intersect. Thus $p^{-1}(f_{t_n}(z_n))=\tilde{f}_{t_n}(x_n)$

converges to $x_{j(i)}=\tilde{f}_t(y_i)$. This is enough to prove continuity since $\tilde{f}_t$ is obviously continuous on $\sigma\times I$ and on $\tau\times I$ for each simplex τ of $p^{-1}(b(K'))$.

The piecewise-linear assertion is clear from the definition of $\tilde{f}_t$ so we shall be finished as soon as we show that $\tilde{f}_t$ is a homeomorphism. But the existence of a unique map shows that $(f_t^{-1})^{\sim}\cdot\tilde{f}_t$ is the identity.

We remark that if σ_i is any simplex of $\tilde{K}$ then the closure of $p^{-1}(C(\sigma, i))$ is equal to $\mathrm{St}(\sigma_i; \tilde{K})$. From this it follows that $\mathrm{st}(\sigma_i; \tilde{K})-b|\tilde{K}|$ is connected, so the singularity function for $\tilde{K}$ is identically one. But then the dimension function is constant on components of $\tilde{K}$. Applying a theorem of Munkres [11, Proposition (5.8)] to the components of $\tilde{K}$ we conclude:

(2.9) *If x in $\tilde{K}$ satisfies $d(x)-i(x)\leqq 2$ then x has a closed $d(x)$-cell neighborhood.*

3. **Approximation theorems.** In this section we shall be proving the approximation theorems that we need. We shall frequently use a metric on $|K|$ called the *linear metric*, and we reserve the symbol λ for this metric. It is defined by the equation

$$\lambda(x, y) = \left(\sum (x_i-y_i)^2\right)^{1/2}$$

where x_i and y_i are the respective barycentric coordinates of x and y. Of course this metric depends on the triangulation K, in particular the distance between points will increase when we subdivide K.

Finally we should mention that many of the approximation results that we prove are true for a general metric space, or at least for a general locally compact metric space. We have tried to write the proofs to bring out this fact, but have stated the theorems for polyhedra since this is all we ever need.

(3.1) *Let $\mathscr{U}$ be a locally finite open cover of $|K|$ by sets with compact closure. Let $c(U)$ be a given positive number for each U in $\mathscr{U}$. Then there exists a positive continuous function $\varepsilon(x)$ on $|K|$ so that $\varepsilon(x)\leqq c(U)$ for all x in U, and all U in $\mathscr{U}$.*

Indeed let $\{f_U\}$ be a partition of unity subordinate to $\mathscr{U}$. For U in $\mathscr{U}$ let $a(U)$ be the minimum of $\{c(V) \mid V\cap U\neq\varnothing,\ V\in\mathscr{U}\}$ (a finite set since $\bar{U}$ is compact). Then $\varepsilon(x)=\sum a(U)f_U(x)$ is the desired function.

We are going to consider approximations to within a positive continuous function. That is, if $f, g\colon |K|\to|L|$ are functions, and if $\varepsilon(x)$ is a positive continuous function on $|K|$, we say that g is an ε-approximation to f if $\rho(f(x), g(x))<\varepsilon(x)$ for all x in $|K|$ where ρ is some preassigned metric on $|L|$. Of course we should speak of an (ε, ρ)-approximation to f, but the next lemma and its corollary show that if we are willing to subdivide L, then $\varepsilon(x)$ may be taken to be a constant and ρ may be taken to be the linear metric of the subdivision.

(3.2) *Let $\varepsilon(x)$ be a positive continuous function on $|K|$, and let ρ be any metric for $|L|$. Then for any continuous map $f\colon |K|\to|L|$ there is a subdivision K_1 of K so that the ρ-diameter of $f(\mathrm{St}(\sigma; K_1))$ is less than the minimum of $\varepsilon(x)$ on $\mathrm{St}(\sigma; K_1)$ for any σ in K_1.*

Case I. $|K|$ is compact. Let $\varepsilon > 0$ be the minimum of $\varepsilon(x)$ on $|K|$. Since

$$f\colon |K| \to |L|$$

is uniformly continuous, there exists $\delta > 0$ so that if $\theta(s, y) < \delta$ then $\rho(f(x), f(y)) < \varepsilon$ where θ is some metric for $|K|$. Choose r so large that the θ-diameter of St $(\sigma; K^{(r)})$ is less than δ. Then $K_1 = K^{(r)}$ satisfies the lemma.

Case II. K is countable. Let $\sigma_1, \sigma_2, \ldots$ be an ordering of K. Let $M_0 = K$, and suppose inductively that we have constructed subdivisions $M_0, M_1, \ldots, M_n$ of K so that for $1 \leqq r \leqq n$,

(i) If σ is in M_r and $\sigma \subset \sigma_1 \cup \cdots \cup \sigma_r$ then the ρ-diameter of $f(\mathrm{St}\,(\sigma; M_r))$ is less than the minimum of $\varepsilon(x)$ on St $(\sigma; M_r)$.

(ii) M_r is a subdivision of M_{r-1} which agrees with M_{r-1} outside

$$\mathrm{St}\,(\mathrm{St}\,(\sigma_r; K); K).$$

The subcomplex St (σ_{n+1}, M_n) is compact so construct the subdivision J of it as in Case I. This can be extended to a subdivision of M_n without subdividing anything outside St $(|J|; M_n)$. This constructs M_{n+1} as desired.

In this process each simplex of K gets subdivided only a finite number of times. Thus if K_1 is the limit of the sequence $M_0, M_1, \ldots$ then K_1 is the desired subdivision.

Case III. K is arbitrary. Then the components of $|K|$ are the polyhedra of countable subcomplexes, which we subdivide by Case II.

Let K be an n-complex, L any complex, ρ any metric for $|L|$, $\varepsilon(x)$ a positive continuous function on $|K|$, and $f\colon |K| \to |L|$ a continuous function. Then there exists a subdivision K_1 of K so that if λ is the linear metric for K_1, and $\lambda(x, y) < (2/(n+1))^{1/2}$, then $\rho(f(x), f(y)) < \varepsilon(x)$.

To see this we suppose first that K_1 is any subdivision of K. Suppose that x and y are points of $|K_1|$ which do not lie in the star of the same vertex of $|K_1|$. Then $\lambda(x, y) = (\sum (x_i - y_i)^2)^{1/2} = (\sum x_i^2 + \sum y_i^2)^{1/2}$ since $x_i \neq 0$ implies $y_i = 0$ and conversely, $y_i \neq 0$ implies $x_i = 0$. But the minimum of $(\sum x_i^2 + \sum y_i^2)^{1/2}$ is taken on when there are $n+1$ nonzero x_i's and $n+1$ nonzero y_i's each equal to $1/(n+1)$. Thus the minimum is $(2/(n+1))^{1/2}$.

Now apply (3.2) to the function f from $|K|$ to $|L|$. If $\lambda(x, y) < (2/(n+1))^{1/2}$ then there is a vertex v so that both x and y are in st $(v; K_1)$. But the ρ-diameter of $f(\mathrm{St}\,(v, K_1))$ is then less than both $\varepsilon(x)$ and $\varepsilon(y)$.

The result we wanted follows easily from this, at least for homeomorphisms. That is

(3.3) *Let h be a homeomorphism of $|K|$ onto $|L|$. Let ρ be a metric on $|L|$ and let $\varepsilon(x)$ be a positive continuous function on $|K|$. Then there is a subdivision L_1 of L, and a number $\delta > 0$ so that if $f\colon |K| \to |L|$ is any continuous function with*

$$\lambda_1(h(x), f(x)) < \delta$$

then $\rho(h(x), f(x)) < \varepsilon(x)$.

In the preceding result use the positive continuous function $\varepsilon \cdot h^{-1}$ and the identity function from L to itself. This gives a subdivision L_1 of L and a number $\delta = (2/(n+1))^{1/2}$, n equal to the dimension of L, so that if $\lambda_1(y, z) < \delta$ then $\rho(y, z) < \varepsilon h^{-1}(y)$. Then if $\lambda_1(h(x), f(x)) < \delta$ then $\rho(h(x), f(x)) < \varepsilon h^{-1}h(x)$.

We come now to a collection of lemmas which say things such as: If homeomorphisms are close, so are their inverses. None of these is particularly surprising but we have not found them in the literature. Most of them are proved by using uniform continuity on the closed stars of vertices, and then quoting (3.1) or (3.2). We shall prove only the first since its proof does not fit this pattern.

(3.4) *Let $f\colon |K| \to |L|$ be a homeomorphism, and let $\varepsilon(x)$ be a positive continuous function on $|L|$. Then there exists a positive continuous function $\delta(x)$ on $|K|$ so that if $g\colon |K| \to |L|$ is a homeomorphism with $\lambda(f(x), g(x)) < \delta(x)$ for all x, then*

$$\lambda(f^{-1}(x), g^{-1}(x)) < \varepsilon(x)$$

for all x, where λ is the appropriate linear metric.

Denote by $B(x; \varepsilon)$ the ball of radius ε about x. If $G(f)$ is the graph of f then we claim

$$U = \bigcup_{x \in |L|} B(f^{-1}(x), \varepsilon(x)) \times \{x\}$$

is a neighborhood of $G(f)$ in $|K| \times |L|$. Indeed let $(f^{-1}(x), x) \in G(f)$. Since $\varepsilon(x)$ is continuous we can choose a neighborhood W of x so that $\varepsilon(y) > \varepsilon(x)/2$ for all y in W. Choose $V \subset W$ so that $\lambda(f^{-1}(x), f^{-1}(y)) < \varepsilon(x)/4$ for all y in V. Then

$$B(f^{-1}(x), \varepsilon(x)/4) \times V$$

is a neighborhood of $(f^{-1}(x), x)$ in $|K| \times |L|$ and is contained in U.

Consider the metric ρ on $|K| \times |L|$ defined by $\rho((x, y), (z, w)) = \lambda(x, z) + \lambda(y, w)$. It is well known (and easily proved) that the function on a metric space which assigns to each point its distance from a fixed set is continuous. Thus the function $\delta(x)$ which is the distance from $(x, f(x))$ to $|K| \times |L| - U$ is a continuous positive function on $|K|$.

If $g\colon |K| \to |L|$ is a homeomorphism with $\lambda(f(x), g(x)) < \delta(x)$ for all x, then $\rho((x, f(x)), (x, g(x))) = \lambda(f(x), g(x)) < \delta(x)$. So $(x, g(x))$ is in U, and $G(g) \subset U$. If $y \in |L|$ then $(g^{-1}(y), y) \in U$, and $|K| \times \{y\} \cap U = B(f^{-1}(y), \varepsilon(y)) \times \{y\}$. Thus $g^{-1}(y) \in B(f^{-1}(y), \varepsilon(y))$.

(3.5) *Let $f\colon |K| \to |L|$ be continuous, and let $\varepsilon(x)$ be a positive continuous function on $|K|$. Then there exists a positive continuous function $\delta(x)$ on $|K|$ so that*

$$\lambda(f(x), f(y)) < \varepsilon(x)$$

whenever $\lambda(x, y) < \delta(x)$.

(3.6) *Let f and $\varepsilon(x)$ be as above. There exists $\delta(x)$ so that if $g\colon |K| \to |L|$ is continuous and $\lambda(f(x), g(x)) < \delta(x)$ for all x, then $\lambda(f(y), g(y)) < \varepsilon(x)$ whenever $\lambda(x, y) < \delta(x)$.*

(3.7) *Let K be an n-complex, and let $X \subset |K|$ have λ-diameter less than $(2/(n+1))^{1/2}$. Then $\bar{X}$ is compact. It follows that a sequence in $|K|$ converges if and only if it is a Cauchy sequence in the linear metric.*

It should be emphasized that (3.7) is only claimed for the linear metric on $|K|$.

These are all the lemmas we need of a general approximation nature. We need one lemma which gives us a hold on the increase in size when we go from a complex to the composition space.

(3.8) *Let K be an n-complex and* $0<\varepsilon<1/(n+1)$. *Let $\alpha: I\to |K|$ be a piecewise-linear arc whose diameter in the linear metric of K' is less than ε. Suppose there exists a "lift" $\tilde{\alpha}: I\to|\tilde{K}|$ of α, that is, $\tilde{\alpha}$ is a piecewise-linear arc for which $p\cdot\tilde{\alpha}=\alpha$. Then the $\tilde{K}$-linear diameter of $\tilde{\alpha}(I)$ is less than $(2n+1)^{1/2}\cdot\varepsilon$.*

There is nothing to prove unless there are points of $\alpha(I)$ having positive barycentric coordinates with respect to a vertex v of K', whose corresponding points on the lift $\tilde{\alpha}(I)$ have positive barycentric coordinates with respect to distinct vertices in $p^{-1}(v)$. The idea of the proof is to show that not too much barycentric weight is concentrated at such points. For convenience in this proof, if z is a point of a complex and v a vertex we denote by $z(v)$ the barycentric coordinate of z with respect to v. We use λ to denote both the linear metric of K' and that of $\tilde{K}$.

If z is on $\alpha(I)$ and v is a vertex of K' with $z(v)\geqq 1/(n+1)$ then $\alpha(I)\subset\operatorname{st}(v; K')$. This is so since if $w(v)=0$ for some point w, then $\lambda(w,z)=(z(v)^2+\text{positive terms})^{1/2}\geqq 1/(n+1)>\varepsilon$. Thus w is not on $\alpha(I)$. Since K is n-dimensional there is a vertex v with $\alpha(I)\subset\operatorname{st}(v; K')$. Let τ be the largest simplex of K' with $\alpha(I)\subset\operatorname{st}(\tau; K')$. If v is a vertex of K' not on τ, and z is a point of $\alpha(I)$ with $z(v)=0$, then for any w on $\alpha(I)$,

$$\text{(a)} \qquad w(v) \leqq \lambda(z, w) < \varepsilon.$$

We claim next that $\operatorname{st}(\tilde{\alpha}(I); \tilde{K})-p^{-1}(b|K|)$ is connected. For let $\tilde{\tau}_1,\ldots,\tilde{\tau}_r$ be the simplexes of $\tilde{K}$ which $\tilde{\alpha}$ meets, redundantly listed in the order in which $\tilde{\alpha}$ meets them. Thus $\tilde{\tau}_i<\tilde{\tau}_{i+1}$ or vice versa. If $\tilde{C}_i=\operatorname{st}(\tilde{\tau}_i,\tilde{K})-p^{-1}|b(K)|$, $C_i=p(\tilde{C}_i)$, and $\tau_i=p(\tilde{\tau}_i)$, then C_i is the component of $\operatorname{st}(\tau_i; K')-|b(K)|$ which determines the simplex $\tilde{\tau}_i$. If $\tilde{\tau}_i<\tilde{\tau}_{i+1}$ then $C_i\supset C_{i+1}$ so $\tilde{C}_i\supset\tilde{C}_{i+1}$. Thus $\bigcup\tilde{C}_i=\operatorname{st}(\tilde{\alpha}(I);\tilde{K})-p^{-1}|b|(K)|$ is connected and mapped by p homeomorphically onto

$$\bigcup C_i = \operatorname{st}(\alpha(I); K')-|b(K)|.$$

Thus $\bigcup C_i$ is connected and hence contained in a single component, say $C(\tau,0)$, of $\operatorname{st}(\tau; K')-|b(K)|$. If τ_0 is the simplex of $p^{-1}(\tau)$ determined by $C(\tau,0)$, then $\tau_0<\tau_i$ for each i since $C(\tau,0)\supset C_i$.

Let $\tilde{v}_0,\ldots,\tilde{v}_k$ be the vertices of τ_0, and let $\tilde{y},\tilde{z}$ be points of $\tilde{\alpha}(I)$. Then

$$\lambda(\tilde{y},\tilde{z}) \leqq \left(\sum_{i=0}^{k}(\tilde{y}(\tilde{v}_i)-\tilde{z}(\tilde{v}_i))^2+\sum\tilde{y}(\tilde{v})^2+\sum\tilde{w}(\tilde{v})^2\right)^{1/2}$$

where the second and third summations are over all other vertices of $\tilde{K}$, and each contains at most $n-k$ nonzero terms. But

$$\varepsilon > \lambda(p(\tilde{y}), p(\tilde{z})) \geqq \left(\sum_{i=0}^{k}(\tilde{y}(\tilde{v}_i)-\tilde{z}(\tilde{v}_i))^2\right)^{1/2}.$$

A term of the form $\tilde{y}(\tilde{v})$ is equal to $p(\tilde{y})(p(\tilde{v}))$ which is less than ε by equation (a) since $p(\tilde{v})$ is not on τ. Putting these estimates together we have

$$\lambda(\tilde{y}, \tilde{z}) < (\varepsilon^2+2(n-k)\varepsilon^2)^{1/2} = (2n-2k+1)^{1/2}\varepsilon \leqq (2n+1)^{1/2}\varepsilon.$$

We introduce some notation. Let us denote by $|K(r)|$ the points of index less than or equal to r. It is the polyhedron of a subcomplex $K(r)$ of K. We denote by $|K(r, m)|$ the points of index r and singularity m. $|K(r, m)|$ is the union of a set $K(r, m)$ of simplexes of K, but is not, in general, closed.

We now prove three approximation lemmas which enable us to move homeomorphisms up to the composition space, and then back down to the base complex.

(3.9) *If K is a complex with composition space $(\tilde{K}, p)$, then the restriction of p to $p^{-1}|K(r, m)|$ is an m-sheeted covering map onto $|K(r, m)|$.*

We show that if σ, τ are in $K'(r, m)$ and $\sigma < \tau$ then each of the m simplexes σ_i in $p^{-1}(\sigma)$ is a face of just one of the m simplexes in $p^{-1}(\tau)$. Then

$$p^{-1}(\mathrm{st}\,(\sigma; K') \cap K'(r, m))$$

has m components each mapped by p homeomorphically onto $\mathrm{st}\,(\sigma; K') \cap K'(r, m)$. But for this we only need to show that each component of $\mathrm{st}\,(\sigma; K')-|b(K)|$ contains exactly one component of $\mathrm{st}\,(\tau; K')-|b(K)|$.

Let x be a point of σ and let $U \subset \mathrm{st}\,(\sigma; K')$ be a triangulated neighborhood of x so that x lies on an r-simplex μ of U. As remarked before the constant value m of the singularity function on μ is the number of components in $\mathrm{st}\,(\mu; U)-|b(K)| = \mathrm{st}\,(\mu; U) \cap (\mathrm{st}\,(\sigma; K')-|b(K)|)$. Since $\mathrm{st}\,(\mu; U)$ is a neighborhood of x in $|K|$, each component of $\mathrm{st}\,(\sigma; K')-|b(K)|$ meets some component of $\mathrm{st}\,(\mu; U)-|b(K)|$, and hence contains exactly one.

Choose n so large that for some r-simplex μ_1 of $U^{(n)}$ $\mathrm{st}\,(\mu_1; U^{(n)}) \subset \mathrm{st}\,(\tau; K')$. This is possible since μ is a neighborhood of x in $|K(r, m)|$. As above each component of $\mathrm{st}\,(\tau; K')-|b(K)|$ contains exactly one component of $\mathrm{st}\,(\mu; U^{(n)})-|b(K)|$. But by (2.5) each component of $\mathrm{st}\,(\mu; U)-|b(K)|$ contains exactly one component of $\mathrm{st}\,(\mu_1; U^{(n)})-|b(K)|$. Thus each component of $\mathrm{st}\,(\sigma; K')-|b(K)|$ meets some component of $\mathrm{st}\,(\tau; K')-|b(K)|$ which it necessarily contains. Since each of these sets has m components, we are done.

Our next lemma shows that we can lift a homeomorphism (even an isotopy) of the singular points of one complex to those of another, at least if it preserves index and singularity with respect to the containing spaces. To do this we need to know that the containing spaces are homeomorphic, and even that we are close to some global homeomorphism in order to make the proof work.

(3.10) *Let $h: |K| \to |L|$ be a homeomorphism of n-complexes and let $\tilde{\varepsilon}(x)$ be a positive continuous function on $p^{-1}|b(K')|$. Then there exists a positive continuous function $\varepsilon(x)$ on $|b(K')|$ so that:*

If $h_1\colon |b(K')| \to |b(L')|$ is a homeomorphism preserving index and singularity with respect to $|K'|$ and $|L'|$, and if $\lambda(h_1(x), h(x)) < \varepsilon(x)$ for all x, then there is a continuous map $\tilde{h}_1\colon p^{-1}|b(K')| \to p^{-1}|b(L')|$ which is an $\tilde{\varepsilon}$-approximation to $\tilde{h}$, and the diagram

$$\begin{array}{ccc} p^{-1}|b(K')| & \xrightarrow{\tilde{h}_1} & p^{-1}|b(L')| \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ |b(K')| & \xrightarrow{h_1} & |b(L')| \end{array}$$

commutes.

If $\tilde{\varepsilon}(x)$ is sufficiently small then $\tilde{h}_1$ is unique and is a homeomorphism. Moreover if there is an isotopy h_t, preserving index and singularity, of $h \mid |b(K')|$ to h_1 with $\lambda(h_t(x), h(x)) < \varepsilon(x)$, then $\tilde{h}_t$ is an isotopy. Finally if h_1 is piecewise-linear on a subcomplex M of $b(K')$, then $\tilde{h}_1$ is piecewise-linear on $p^{-1}(M)$.

We call a function which makes the diagram commute a *lift* of h_1. The proof proceeds as follows: In Step (1) we show that for $\tilde{\varepsilon}(x)$ sufficiently small, a continuous lift which is an $\tilde{\varepsilon}(x)$-approximation to $\tilde{h}$ is unique. In the course of this we get a formula for $\tilde{h}_1$. In Step (2) we show that for $\varepsilon(x)$ sufficiently small the formula of Step (1) does define a lift. In Step (3) we show that the lift of Step (2) is an $\tilde{\varepsilon}(x)$-approximation to $\tilde{h}$. In Step (4) we show that for $\varepsilon(x)$ sufficiently small, the lift $\tilde{h}_t(x)$ is continuous in (x, t). In Step (5) we use (3.4) together with the usual categorical argument to show that $\tilde{h}$ is a homeomorphism. The piecewise-linear part is pointed out at the end.

Step (1). Let σ be a simplex of $K'(r, m)$. Then if h_1 is a homeomorphism preserving index and singularity we have $h_1(\sigma) \subset |L(r, m)|$. By (3.9) $p^{-1}h_1(\sigma)$ consists of m components each mapped by p homeomorphically onto $h_1(\sigma)$. It follows that if σ_i is a simplex of $p^{-1}(\sigma)$ then for some component $C(i)$ of $p^{-1}h_1(\sigma)$, the map $\tilde{h}_1|\sigma_i$ is given by

$$\tilde{h}_1|\sigma_i = (p|C(i))^{-1}h_1(p|\sigma_i). \tag{1}$$

Thus we shall have shown uniqueness of a continuous lift $\tilde{h}_1$ of h_1 as soon as we show that for $\tilde{\varepsilon}(x)$ sufficiently small, there is only one possible choice for $C(i)$.

Let us suppose that for each σ_i in $p^{-1}(\sigma)$, $\tilde{\varepsilon}(\sigma_i')$ is less than the distance from $\tilde{h}(\sigma_i')$ to $p^{-1}(|b(L)|) - \tilde{h}(\mathrm{st}\,(\sigma_i))$. If $\tilde{h}_1$ is an $\tilde{\varepsilon}(x)$-approximation to $\tilde{h}$, then $\tilde{h}_1(\sigma_i')$ is in $\tilde{h}(\mathrm{st}\,(\sigma_i))$. We show that $p^{-1}(h_1(\sigma')) \cap \tilde{h}(\mathrm{st}\,(\sigma_i))$ is at most one point, and this selects $C(i)$ for us. But $p^{-1}(h_1(\sigma')) \cap \tilde{h}(\mathrm{st}\,(\sigma_i))$ is contained in $p^{-1}[|L(r, m)| \cap h(\mathrm{st}\,(\sigma))]$, and by (3.9) again this consists of m components, each mapped by p homeomorphically onto $|L(r, m)| \cap h(\mathrm{st}\,(\sigma))$. Then each of these m components contains at most one of the m points of $p^{-1}(h_1(\sigma'))$.

Step (2). We can use formula (1) to define a lift of h_1 provided $h_1(\sigma')$ is in $h(\mathrm{st}\,(\sigma))$, for then each component of $p^{-1}[|L(r, m)| \cap \tilde{h}(\mathrm{st}\,(\sigma))]$ contains exactly one of the points of $p^{-1}h_1(\sigma')$. But $h_1(\sigma')$ is in $h(\mathrm{st}\,(\sigma))$ as long as $\varepsilon(\sigma')$ is less than the distance from $h(\sigma')$ to $|L'|-h(\mathrm{st}\,(\sigma))$. Assign to each of the open subsets $\mathrm{st}\,(\sigma; b(K'))$ of $|b(K')|$ the above distance. Then (3.1) shows that a sufficiently small $\varepsilon(x)$ exists.

Step (3). We assume that $\varepsilon(x)$ is so small that

(i) $\varepsilon(x)$ is less than the minimum of $\tilde{\varepsilon}(y)/4(2n+1)^{1/2}$ over the compact set $p^{-1}(\mathrm{St}\,(\sigma))$ for x in σ and any σ in $b(K')$.

(ii) $\varepsilon(x)<1/(n+1)$.

Both may be achieved by using (3.1). We apply (3.2) to find a subdivision K_1 of $b(K')$ so fine that the diameter of $h(\mathrm{St}\,(\sigma; K_1))$ is less than the minimum on $\mathrm{St}\,(\sigma; K_1)$ of some function satisfying (i) and (ii). We assume in addition that $\varepsilon(x)$ satisfies

(iii) $\varepsilon(\sigma')$ is less than the distance from $h(\sigma')$ to $|b(L')|-h(\mathrm{st}\,(\sigma))$ for each σ in K_1.

Let $h_1\colon |b(K)| \to |b(L)|$ be a homeomorphism preserving index and singularity which is an $\varepsilon(x)$-approximation to $h \mid |b(K)|$. By Steps (1) and (2) there exists a unique lift $\tilde{h}_1$ of h_1 to $p^{-1}|K_1|$ which satisfies formula (1) with respect to simplexes σ of K_1. We claim that $\tilde{h}_1$ is an $\tilde{\varepsilon}(x)$-approximation to $\tilde{h}$.

Let $\sigma\subset|K(r,m)|$ be a simplex of K_1, σ_i a simplex of $p^{-1}(\sigma)$, x a point of σ_i and $y=p(x)$. By choice of K_1, the diameter of $h(\mathrm{st}\,(\sigma)\cap|K(r,m)|)$ is less than $\tilde{\varepsilon}(x)/4(2n+1)^{1/2}$. Using the triangle inequality, and the fact that h_1 is an $\varepsilon(x)$-approximation to h we have that the diameter of $h_1(\mathrm{st}\,(\sigma)\cap|K(r,m)|)$ is less than

$$3\tilde{\varepsilon}(x)/4(2n+1)^{1/2}.$$

Now the set $h(\mathrm{st}\,(\sigma)\cap|K(r,m)|)\cap h_1(\mathrm{st}\,(\sigma)\cap|K(r,m)|)$ is not empty since it contains $h(\sigma')$ by (iii). Thus

$$X = h(\mathrm{st}\,(\sigma)\cap|K(r,m)|)\cup h_1(\mathrm{st}\,(\sigma)\cap|K(r,m)|)$$

is a connected open subset of $|L(r,m)|$ of diameter less than $\tilde{\varepsilon}(x)/(2n+1)^{1/2}$.

Choose a piecewise-linear arc $\alpha(t)$ in X joining $h(y)=\alpha(0)$ to $h_1(y)=\alpha(1)$, and so that $\alpha(1/2)=h(\sigma')$. Since p is a covering map from $p^{-1}|L(r,m)|$ to $|L(r,m)|$, the arc $\alpha(t)$ has a unique continuous lift $\tilde{\alpha}(t)$ to $p^{-1}|L(r,m)|$ starting at $\tilde{h}(x)$. Then $\tilde{\alpha}(1/2)=\tilde{h}(\sigma_i')$, so by formula (1) $\tilde{\alpha}(1)=\tilde{h}_1(x)$. Since the diameter of $\alpha(I)$ is less than $\tilde{\varepsilon}(x)/(2n+1)^{1/2}$ and less than $1/(n+1)$, we may use (3.8) to conclude that the diameter of $\tilde{\alpha}(I)$ is less than $\tilde{\varepsilon}(x)$. Thus $\lambda(\tilde{h}(x),\tilde{h}_1(x))=\lambda(\tilde{\alpha}(0),\tilde{\alpha}(1))<\tilde{\varepsilon}(x)$.

Step (4). We assume now that an isotopy h_t preserving index and singularity is given so that $\lambda(h_t(x),h(x))<\varepsilon(x)$ for x in $|b(K)|$, and we show that $(x,t)\to\tilde{h}_t(x)$ is continuous for $\varepsilon(x)$ sufficiently small. From formula (1) it follows that $(x,t)\to\tilde{h}_t(x)$ is continuous on $\sigma_i\times I$ for each simplex σ of $b(K')$. We shall show that if $\sigma_i<\tau_j$ that this map is continuous on $(\sigma_i\cup\tau_j)\times I$. From this it follows quickly that the map is globally continuous. A specialization of these arguments will show that $\tilde{h}_1$ is continuous even when we do not assume the existence of an isotopy.

We shall assume that $\varepsilon(x)$ satisfies all previous conditions of smallness (so unique lifts exist which are $\tilde{\varepsilon}(x)$-approximations to $\tilde{h}$) and we shall develop the further smallness conditions as we need them. The proof in this step is in two parts. We show first that if $\{x_q\}$ is a sequence in τ_j converging to σ_i', then $\tilde{h}_1(x_q)$ converges to $\tilde{h}_1(\sigma_i')$. We then use this to prove the continuity.

The sequence $\{\tilde{h}(x_q)\}$ converges to $\tilde{h}(\sigma_i')$ so it is Cauchy. Thus for k and q sufficiently large we have $\lambda(\tilde{h}(x_k), \tilde{h}(x_q)) < \varepsilon(\sigma')$. Since h and h_1 are continuous we may also have $\lambda(hp(x_k), h_1p(x_k)) < \varepsilon(\sigma')$ and $\lambda(hp(x_q), h_1p(x_q)) < \varepsilon(\sigma')$. As seen in Step (3) it follows that $\lambda(\tilde{h}(x_k), \tilde{h}_1(x_k)) < (2n+1)^{1/2}\varepsilon(\sigma')$ and similarly for x_q. By the triangle inequality then $\lambda(\tilde{h}_1(x_k), \tilde{h}_1(x_q)) < 3(2n+1)^{1/2}\varepsilon(\sigma')$. We may assume $\varepsilon(x)$ small enough so that this is less than the minimum distance between points in $p^{-1}h_1(\sigma')$. Since the points $\{\tilde{h}_1(x_q)\}$ all approach points of $p^{-1}(h_1(\sigma'))$, it follows that they converge to a unique point z of $p^{-1}(h_1(\sigma'))$.

We must show that $z = \tilde{h}_1(\sigma_i')$. But for q large if we add up the distances from z to $\tilde{h}_1(x_q)$, to $\tilde{h}(x_q)$, to $\tilde{h}(\sigma_i')$, to $\tilde{h}_1(\sigma_i')$ we get a number less than $4(2n+1)^{1/2}\varepsilon(\sigma')$. Since this may also be assumed less than the minimum distance between points in $p^{-1}h_1(\sigma')$ we conclude that $z = \tilde{h}_1(\sigma')$.

Now let $\{(y_q, t_q)\}$ be a sequence in $\tau_j \times I$ converging to a point (y, t) of $\sigma_i \times I$. The sequence $\{h_{t_q}(p(y_q))\}$ converges to $h_t(p(y))$, hence it is Cauchy. As above it follows that $\tilde{h}_{t_q}(y_q)$ is Cauchy and converges to a point of $p^{-1}(h_t(p(y)))$. If $\{(z_q, s_q)\}$ is another sequence in $\tau_j \times I$ converging to (y, t), then the sequence $\ldots(y_q, t_q)$, $(z_q, s_q), (y_{q+1}, t_{q+1}), \ldots$ also converges to (y, t). It follows that $\ldots, \tilde{h}_{t_q}(y_q)$, $\tilde{h}_{s_q}(z_q)\ldots$ converges, so that $\{\tilde{h}_{t_q}(y_q)\}$ and $\{\tilde{h}_{s_q}(z_q)\}$ converge to the same point of $p^{-1}(p(y))$ which we denote by $f(y, t)$. It is a standard diagonal sequence argument to show that f is a continuous map on $\sigma_i \times I$.

The map $(x, t) \to h_t(x)$ defined on $\sigma \times I$ now has two continuous lifts, $(x, t) \to f(x, t)$ and $(x, t) \to \tilde{h}_t(x)$. Since p is a covering map on $p^{-1}|L(r, m)|$ we will know these maps are identical if they agree at one point. But in the first half of this step we showed that $\tilde{h}_1(\sigma_i') = f(\sigma_i', 1)$. We conclude that $\{\tilde{h}_{t_q}(y_q)\}$ converges to $\tilde{h}_t(y)$. From this follows continuity on $(\sigma_i \cup \tau_j) \times I$ and Step (4) is done.

Step (5). We claim now that if $\varepsilon(x)$ is sufficiently small then $\tilde{h}_t$ is a homeomorphism. To see this we note that by Lemma (3.4) we can make h_t^{-1} as close to h^{-1} as we wish, by making $\varepsilon(x)$ small. Thus we can insure the existence of a unique continuous lift $(h_t^{-1})^{\sim}$ of h_t^{-1}, as close to $(h^{-1})^{\sim} = (\tilde{h})^{-1}$ as we wish. Then $(\tilde{h}_t)\cdot(h_t^{-1})^{\sim}$ is a lift of the identity function, which by (3.5) and (3.6) will be as close to $\tilde{h}\cdot(\tilde{h})^{-1}$ as we wish.

The last lemma told us when we could lift maps from a complex up to the composition space. The next step is to show when they can be brought back down. It is the obvious answer.

(3.11) *Let* $\tilde{h}\colon |\tilde{K}| \to |\tilde{L}|$ *and* $k\colon |bK| \to |bL|$ *be continuous maps so that the diagram*

$$\begin{array}{ccc} p^{-1}|bK| & \xrightarrow{\tilde{h}} & |\tilde{L}| \\ \Big\downarrow{\scriptstyle p} & & \Big\downarrow{\scriptstyle p} \\ |bK| & \xrightarrow{k} & |bL| \end{array}$$

commutes. Then there exists a continuous map $h\colon |K| \to |L|$ which extends k and such that the diagram

$$\begin{array}{ccc} |\tilde{K}| & \xrightarrow{\tilde{h}} & |\tilde{L}| \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ |K| & \xrightarrow{h} & |L| \end{array}$$

commutes. The map h is unique. If $\tilde{h}$ and k are piecewise-linear so is h. If h and k are homeomorphisms so is h, and if $\tilde{h}$ and k vary by an isotopy so does h.

In order to have the second diagram commute we must define

$$h \mid |K| - |bK| = p\tilde{h}p^{-1} \mid |K| - |bK|.$$

Since $|K| - |bK|$ is dense in $|K|$ this proves the uniqueness, and defines h on $|K| - |bK|$. We define h on $|bK|$ to be k. Then the piecewise-linear assertion is clear.

We claim that h is continuous. Let σ be a simplex of $K' - bK'$ with a face τ in bK'. Then there is a simplex τ_i in $p^{-1}\tau$ which has $\sigma_1 = p^{-1}\sigma$ as a face. Then $h|(\sigma \cup \tau) = p\tilde{h}(p|\sigma_1 \cup \tau_i)^{-1}$ which is continuous. Thus $h|\bar{\sigma}$ is continuous for each σ in $K' - bK'$. Since this gives a locally finite closed cover of $|K|$ we have shown that h is continuous.

It follows quickly from the uniqueness that if $\tilde{h}^{-1}$ and k^{-1} exist then the map they define is h^{-1}. Finally if $\tilde{h}$ and k vary by an isotopy then the continuity argument applied to $\bar{\sigma} \times I$ (as in (3.10)) shows that h varies continuously.

4. **The Hauptvermutung.** In this section we shall establish a strong form of the Hauptvermutung for complexes of dimension less than four. We begin with three lemmas whose proofs we leave to the reader.

(4.1) *Let L be a subcomplex of K, and L_1 a subdivision of L. Then there is a subdivision K_1 of K which agrees with L_1 on $|L|$, and agrees with K outside* st $(|L|; K)$.

(4.2) *Let L be a* 1*-dimensional subcomplex of K, and let $A \subset |L|$ be an isolated set of points (that is, A has no limit points in $|K|$). Then there is a subdivision K_1 of K whose vertices are the vertices of K together with the points of A.*

(4.3) *Let $\bar{\sigma}, \bar{\tau}$ be closed* 1*-simplexes. Let $f\colon \bar{\sigma} \to \bar{\tau}$ be a homeomorphism, and let ε be a positive number. Then there exists an isotopy g_t so that $g_0 = f$, g_1 is piecewise-linear, and $\lambda(f(x), g_t(x)) < \varepsilon$ for all $t \in I$ and x in $\bar{\sigma}$.*

The next proposition provides, in particular, a strong form of the Hauptvermutung for 1-dimensional complexes.

(4.4) *Let $f\colon |K| \to |L|$ be a homeomorphism. Let K_1 and L_1 be* 1*-dimensional subcomplexes of K and L respectively and suppose $f(|K_1|) = |L_1|$. Suppose finally that f is a PL map on a possibly empty subcomplex K_2 of K. Then for any* *positive continuous function* $\varepsilon(x)$ *on* $|K|$ *there exists an isotopy* $g_t\colon |K| \to |L|$ *so that*

(i) $g_0 = f$ *and* $g_t \mid |K_2| = f \mid |K_2|$.

(ii) $g_t(|K_1|) = |L_1|$ *and* $g_1 \mid |K_1|$ *is piecewise-linear.*

(iii) $\lambda(f(x), g_t(x)) < \varepsilon(x)$ *for all* $x \in |K|$ *and* $t \in I$.

(iv) $g_t \mid |K| - \operatorname{st}(|K_1|; K) = f \mid |K| - \operatorname{st}(|K_1|; K)$.

The idea is to deform f to be piecewise-linear on K_1, and extend the maps of the deformation to all of $|K|$ by joining them back to f. In order for this to be possible we must be careful that the maps of the deformation preserve index and singularity with respect to $|K|$ and not just with respect to $|K_1|$.

Let A_K be f^{-1} applied to the vertices of L_1 and let A_L be f applied to the vertices of K_1. Then $A_K \subset |K_1|$, $A_L \subset |L_1|$ and they are both isolated sets since f is a homeomorphism. According to (4.2) there are subdivisions of K and L which change the sets of vertices by adding A_K and A_L respectively to them. So as not to complicate the notation we shall denote the subdivisions of all the complexes by the same symbols as before. It follows then that f maps the vertices of K_1 one-to-one onto the vertices of L_1 since we introduced no other new vertices. We shall also assume that K and L are barycentric subdivisions.

We claim that f maps simplexes of K_1 homeomorphically onto simplexes of L_1. If σ is a 1-simplex of K_1 then $f(\sigma)$ is open in $|L_1|$ and contains no vertices of L_1. Thus $f(\sigma)$ is a union of 1-simplexes of L_1 and since it is connected, $f(\sigma)$ is a single 1-simplex of L_1.

For any 1-simplex σ of K_1 we let $\varepsilon(\sigma) = \min\{\varepsilon(x) \mid x \in \operatorname{St}(\sigma; K)\}$. Choose $\delta(\sigma) > 0$ so that if $x, y \in \operatorname{St}(\sigma; K)$ and $\lambda(x, y) < \delta(\sigma)$ then $\lambda(f(x), f(y)) < \varepsilon(\sigma)$. Notice that if $h\colon \operatorname{St}(\sigma; K) \to \operatorname{St}(\sigma; K)$ moves no point by more than $\delta(\sigma)$, then

$$\lambda(f(h(x)), f(x)) < \varepsilon(x)$$

for each x in $\operatorname{St}(\sigma; K)$.

For each 1-simplex σ of K_1 we apply (4.3) to the map $f|\bar{\sigma}$. This gives us an isotopy $g_t\colon |K_1| \to |L_1|$ so that g_t agrees with f on the vertices, g_1 is piecewise-linear. We assume that g_t is chosen so close to f that $\lambda(x, f^{-1}g_t(x)) < \delta(\sigma)$ for any $x \in \sigma$ and all $t \in I$. We agree to let $g_t \mid |K_1 \cap K_2| = f \mid |K_1 \cap K_2|$ since f is already piecewise-linear there.

We let $h_t = f^{-1} \cdot g_t$, an isotopy of $|K_1|$ which keeps simplexes of K_1 on themselves, and so that h_0 is the identity map. We shall extend h_t to an isotopy of $|K|$. We note first that since we have barycentric subdivisions, K_1 is full in K (that is, each simplex of K having all its vertices in K_1, lies in K). It follows that each simplex of $\operatorname{st}(K_1; K)$ is uniquely the join of a simplex of $\operatorname{St}(K_1; K) - \operatorname{st}(K_1; K)$ with a simplex of K_1, or else lies in K_1.

From this we may extend h_t to $\operatorname{St}(K_1; K)$ by joining it to the identity map on $\operatorname{St}(K_1; K) - \operatorname{st}(K_1; K)$ and then to all of K by making it the identity map outside of $\operatorname{St}(K_1; K)$. Then $f \cdot h_t$ extends g_t to all of $|K|$, and we claim that this is the desired isotopy.

We note first that conditions (i), (ii), and (iv) are clearly satisfied. As for condition (iii) we need examine only points in st $(K_1; K)$. So let σ be a 1-simplex of K_1, and let τ be a simplex of St $(K_1; K) -$ st $(K_1; K)$ so that $\sigma * \tau$ is in st $(\sigma; K)$. Then by construction, $\lambda(x, h_t(x)) < \delta(\sigma)$ for any x in $\sigma * \tau$. But then

$$\lambda(f(x), g_t(x)) < \varepsilon(\sigma) \leqq \varepsilon(x).$$

The next lemma is a form of the Hauptvermutung for 2-manifolds. We need the special form here to prove the Hauptvermutung for 2-complexes.

(4.5) *Let M and N be triangulated 2-manifolds with boundary. Let A be a set of isolated points in M, let $\varepsilon(x)$ be a positive continuous function on M, and let $f\colon M \to N$ be a homeomorphism which is a PL map from ∂M to ∂N. Then there is an isotopy g_t from M to N so that*

(i) $g_t | \partial M \cup A = f | \partial M \cup A$,

(ii) $\lambda(g_t(x), f(x)) < \varepsilon(x)$ *all* $x \in M$,

(iii) g_1 *is piecewise-linear.*

The proof will be in several steps, each one being an isotopy which "improves" f without disturbing previous improvements. Since isotopy is a transitive notion, after each step we shall simply assume that f has all the improvements to date. We may assume $A \cap \partial M = \varnothing$.

The first step is an isotopy of f to a homeomorphism which is piecewise-linear on a neighborhood of A. Let $a \in A$ and let D be a small polyhedral cell neighborhood of $f(a)$. Choose a polyhedral cell neighborhood C of a so that $f(C) \subset \operatorname{int}(D)$. Finally choose a polyhedral cell neighborhood D' of $f(a)$ with $D' \subset \operatorname{int}(f(C)) = f(\operatorname{int}(C))$. Since the region $D - \operatorname{int}(D')$ is piecewise linearly homeomorphic to $\partial D \times [0, 1]$ we can perform an isotopy on this region which does not move points of $\partial D \cup \partial D'$, and which carries $f(\partial C)$ to the boundary of another polyhedral cell neighborhood D'' of $f(a)$. We may even assume that the composition of f with the end of the isotopy is piecewise-linear on ∂C. Thus we may as well assume that $f(C) = D''$ and $f | \partial C$ is piecewise-linear.

Now consider a homeomorphism of a 2-cell to itself which is the identity on one interior point, and on the boundary of the 2-cell. Then one easily proves that the homeomorphism is isotopic to the identity function via an isotopy which is the identity on the boundary and on the interior point.

We apply this to our situation as follows. The disk D'' has two triangulations, the one from N, and the one induced from C by the map f. We may as well assume that these triangulations are maps from a standard triangle onto D'', which agree on the boundary of the triangle and send its barycenter onto $f(a)$. This gives a homeomorphism of D'' to itself which is the identity on the boundary of D'' and on $f(a)$. An isotopy of this homeomorphism to the identity, which leaves $\partial D''$ and $f(a)$ fixed, will isotope $f | C$ to a piecewise-linear homeomorphism from C onto D'' without changing f on ∂C or on a.

Since D chosen at the beginning was as small as we liked, we may do this isotopy simultaneously for all the points of A by choosing the D's disjoint. Moreover if the D's are small, the isotopy moves points as little as we like.

We begin the next step by punching out the interiors of the C's from M, and the interiors of the $f(C)$'s from N. This leaves us with new manifolds which we still denote M and N, and we have eliminated the set A completely. We proceed in this second step to make f piecewise-linear in a neighborhood of ∂M, in very much the same way as the first step.

Choose regular neighborhoods of ∂M and ∂N in M and N respectively which are piecewise linearly homeomorphic to $\partial M \times [0, 2]$ and $\partial N \times [0, 2]$. Define an isotopy g_t from M to N as follows: Stretch $M - \partial M \times [0, t]$ over M without moving points outside $\partial M \times [0, 2]$, then map by f, and then shrink N to $N - \partial N \times [0, t]$. Map $\partial M \times [0, t]$ onto $\partial N \times [0, t]$ by $(f | \partial M) \times \mathrm{id}$. This isotopes f to the homeomorphism g_1 which is piecewise-linear from $\partial M \times [0, 1]$ onto $\partial N \times [0, 1]$. This isotopy will be an $\varepsilon(x)$-approximation to $f(x)$, if the regular neighborhoods $\partial M \times [0, 2]$ and $\partial N \times [0, 2]$ are chosen small enough.

In this third and final step we assume that f maps the regular neighborhood $\partial M \times [0, 1]$ piecewise linearly onto the regular neighborhood $\partial N \times [0, 1]$, and maps $\{m\} \times [0, 1]$ linearly onto $\{f(m)\} \times [0, 1]$. It is known [1] that a homeomorphism between 2-manifolds may be approximated arbitrarily closely by a piecewise-linear homeomorphism. Furthermore Kister [7] has shown that there is an isotopy moving points arbitrarily little between any two sufficiently close homeomorphisms of a 2-manifold. The second of these approximations is to within a constant distance in the linear metric. But we may apply (3.3) to get a subdivision of N so fine that if $\lambda(n, n') < (2/3)^{1/2}$ then $\rho(n, n') < \varepsilon(f^{-1}(n))$ where ρ is the metric we started with on N, and λ is the linear metric of the subdivision.

We apply these two theorems to the homeomorphism f, getting first a piecewise-linear homeomorphism which agrees with f on $\partial M \times [0, 1]$, and then an isotopy of this homeomorphism to f every stage agreeing with f on $\partial M \times [0, \frac{1}{2}]$ (this last uses Corollary 2 of [7]). This completes the proof.

We shall now prove the Hauptvermutung for 2-complexes.

(4.6) THEOREM. *Let K and L be 2-complexes, and let $f\colon |K| \to |L|$ be a homeomorphism. Suppose that f is PL on a possibly empty subcomplex M of K. Finally let $\varepsilon(x)$ be a positive continuous function on $|K|$. Then there is an isotopy $g_t\colon |K| \to |L|$ so that*

(i) $g_0 = f$ *and* $g_t \mid |M| = f \mid |M|$,

(ii) g_1 *is piecewise-linear*,

(iii) $\rho(g_t(x), f(x)) < \varepsilon(x)$ *for all x in* $|K|$.

Proof. The proof proceeds roughly as follows: We make f piecewise-linear on the singular set. Then we tear apart K to make it a 2-manifold. We isotope the lift of f to become piecewise-linear, being careful that we do not change f on the lift of the singular set. Finally we push the isotopy back down to $|K|$ and we are done. Of course we have the problem of keeping track of M in all this, so we shall first simplify M as much as possible.

Our first claim is that we can eliminate int $|M|$. For if the theorem is true with respect to $f \mid |K|-\text{int}\,|M|$, $|K|-\text{int}\,|M|$, and $|M|-\text{int}\,|M|$, then letting the isotopy be constantly f on int $|M|$ (as we must) we find that the theorem is true generally. So we shall assume $\text{int}\,(M)=\varnothing$, in particular M is at most 1-dimensional. We shall assume further that K and L have been subdivided so that f is simplicial on M.

If σ is a 1-simplex of M not in bK, let us adjoin $\bar{\sigma} * v$ to K, and $f(\bar{\sigma}) * v$ to L where v is a new point. Extend f to $\bar{\sigma} * v$ by $f *$ identity, and extend $\varepsilon(x)$ to $\bar{\sigma} * v$ to map each ray $q * v$ linearly onto the interval $[\varepsilon(q), \frac{1}{2}]$ of real numbers, sending v to $1/2$. Doing this for each such σ we arrive at new complexes K_1, L_1 and new maps $f_1(x)$, $\varepsilon_1(x)$. The only points of M not in bK_1 are vertices. If h_t is an isotopy of f_1 as in the theorem, then for each v as above, $\lambda(f_1(v), h_t(v))=\lambda(v, h_t(v))<1/2$. Thus $h_t(v)$ is in $f(\bar{\sigma}) * v$. But $h_t \mid |M|=f \mid |M|$ so h_t maps $(\bar{\sigma} * v)-\bar{\sigma}$ onto $f(\bar{\sigma}) * v-\bar{\sigma}$. Then $g_t=h_t \mid |K|$ satisfies all conditions of the theorem. We may then assume that the only points of M not in bK are vertices.

Let K_1 be the subcomplex of K consisting of all points of index less than two. By (4.4) there is an isotopy h_t of f to a homeomorphism h_1 which is piecewise-linear on K_1, each h_t agrees with f on M, and is an $\varepsilon(x)$-approximation to f. For this reason we may assume that f itself is piecewise-linear on K_1.

To finish the proof we let $(\tilde{K}, p)$ be the composition space, and let N be the subcomplex of $\tilde{K}$ of constant local dimension two. By (2.9), N is a 2-manifold with boundary. Notice that the boundary ∂N of N is contained in $p^{-1}(bK) \cap N$. Thus $p^{-1}(bK \cup M) \cap N=\partial N \cup A$ where A is a set of vertices interior to N. We let $\tilde{f}$ be the lift of f to $\tilde{K}$. By (4.5) we can find an isotopy $\tilde{g}_t$ of $\tilde{f}|N$ to a piecewise-linear homeomorphism, each $\tilde{g}_t$ agreeing with $\tilde{f}$ on $\partial N \cup A$. Moreover we may require that $\tilde{\lambda}(\tilde{g}_t(x), \tilde{f}(x))<\varepsilon(p(x))$ for each x in N where $\tilde{\lambda}$ is the linear metric on $\tilde{L}$. We extend the $\tilde{g}_t$ to the rest of $\tilde{K}$ by letting them equal $\tilde{f}$ there. Now by (3.10) there is a unique isotopy g_t so that the diagram

$$\begin{tikzcd} |\tilde{K}| \arrow[r, "\tilde{g}_t"] \arrow[d, "p"'] & |\tilde{L}| \arrow[d, "p"] \\ |K| \arrow[r, "g_t"] & |L| \end{tikzcd}$$

commutes. Since $\tilde{g}_1$ is piecewise-linear so is g_1, and since $\tilde{g}_t$ agrees with $\tilde{f}$ on $p^{-1}(K_1 \cup M)$, g_t agrees with f on $K_1 \cup M$. Finally let $x \in |K|$ and let $p(y)=x$. Then $\tilde{\lambda}(\tilde{f}(y), \tilde{g}_t(y))<\varepsilon(x)$ hence $\lambda(f(x), g_t(x))<\varepsilon(x)$. This last inequality proves the theorem.

In the next lemma we shall be working with the cone over a 2-manifold M. This has the form $M * v$ where v is a vertex not in M. Now we shall think of $M * v$

as $M \times [0, 1]$ with $M \times \{0\}$ identified to the point v. This notation will be used without further comment. We shall also use $t \cdot (m, s)$ for the point $(m, t \cdot s)$.

(4.7) *Let M be a compact connected 2-manifold and let $f\colon M * v \to M * v$ be an embedding onto a neighborhood of v. Let A be an isolated set of points in M, let $0 < a < 1$ and suppose $f(m, t) = a \cdot (m, t)$ for $m \in \partial M \cup A$, $0 \leqq t \leqq 1$. Then there is an isotopy f_t and a number b, $0 < b < 1$ so that*

(i) *$f_t | M \times \{1\} \cup (\partial M \cup A) * v = f | M \times \{1\} \cup (\partial M \cup A) * v$.*

(ii) *f_0 maps $M \times \{b\}$ linearly to $M \times \{a \cdot b\}$.*

(iii) *$f_0(m, t) = t/b \cdot f_0(m, b)$ for $0 \leqq t \leqq b$.*

If M is a 2-sphere and A is vacuous, it is possible that $f(v) \neq v$. However, a small isotopy will allow us to assume this. In all other cases it follows from the hypotheses that $f(v) = v$.

Consider the closed region R in the range space bounded by $f(M \times \{1\})$ and $M \times \{a \cdot b\}$, where $b < 1$ is chosen so that these surfaces are disjoint. We claim there exists a homeomorphism h of $M \times [b, 1]$ onto R so that h agrees with f on the set $M \times \{1\} \cup (\partial M \cup A) \times [b, 1]$. To prove this we shall apply Theorems (7.2) and (6.1) of [3]. We state the theorems here for convenience. Let us note that an embedding of one manifold into another is regular if it sends boundary into boundary, and interior into interior.

THEOREM ((6.1) OF [3]). *Let M be a compact connected polyhedral 2-manifold. Let α_j, $j = 1, \ldots, n$ be a finite number of pairwise disjoint simple arcs piecewise linearly and regularly embedded in $M \times [0, 1]$. We suppose that $\alpha_j(i) \in \operatorname{int}(M \times \{i\})$ for $i = 0, 1$. If the homomorphism induced by inclusion*

$$\pi_1\left(M \times \{0\} - \bigcup_{j=1}^{n} \alpha_j(0)\right) \to \pi_1\left(M \times [0, 1] - \bigcup_{j=1}^{n} \alpha_j([0, 1])\right)$$

is onto, then there is a piecewise-linear homeomorphism h of $M \times [0, 1]$ onto itself such that

$$\begin{aligned} h(p, 0) &= (p, 0) \quad \text{for } p \in M \\ h(p, t) &= (p, t) \quad \text{for } p \in \partial(M),\ t \in [0, 1] \\ h(\alpha_j(t)) &= (p_j, t) \quad \text{where } \alpha_j(0) = (p_j, 0), \qquad j = 1, \ldots, n,\ t \in [0, 1]. \end{aligned}$$

THEOREM ((7.2) OF [3]). *Let M be a compact polyhedral 2-manifold. Let*

$$f\colon M \to M \times [0, 1]$$

be a regular piecewise-linear embedding such that $f(M)$ separates $M \times \{0\}$ from $M \times \{1\}$. Then there exists a piecewise-linear homeomorphism h of $M \times [0, 1]$ onto itself so that $hf(m) = (m, \frac{1}{2})$ for all $m \in M$.

In order to apply these theorems we must show that all maps involved may be assumed piecewise-linear.

To show piecewise linearity we note that R is a 3-manifold whose boundary is triangulated by f on $f(M\times\{1\})$, and by the polyhedral structure of $M * \{v\}$ on the rest. By [9] this triangulation may be extended to R. If c is chosen so that $f^{-1}(M\times\{a\cdot b\})$ lies above $M\times\{c\}$, then the above trick triangulates the region $M\times[c, 1]$ in such a way that f^{-1} is a piecewise-linear embedding of $M\times\{a\cdot b\}$ into this region separating $M\times\{1\}$ from $M\times\{c\}$. According to (7.2) of [3] the two halves into which $f^{-1}(M\times\{a\cdot b\})$ splits $M\times[c, 1]$ are both products, and the upper one of these is homeomorphic to R.

To apply the Theorem (6.1) of [3] we must know that $A\times[c, 1]$ is tame in this different triangulation, but comparing the embeddings in the domain and range spaces it is trivial to see that $A\times[c, 1]$ is locally tame. By [1] it is tame. To show that h may be taken to agree with f on $A\times[b, 1]$ we must show that $f(A\times[b, 1])$ is unknotted in the produce structure on R. For this it is enough to show that the inclusion map induces an epimorphism of $\pi_1(f((M-A)\times\{1\}))$ to $\pi_1(R-(A\times[0, 1]))$. Again we play off domain and range against each other. Note that

$$\pi_1((M-A)\times\{a\cdot b\})\to\pi_1(R-(A\times[0, 1]))$$

is a monomorphism since its composition with

$$\pi_1(R-(A\times[0, 1]))\to\pi_1((M-A)\times[0, 1])$$

is a monomorphism. Thus $\pi_1((M-A)\times[c, 1])$ is the free product with amalgamation over $\pi_1(f^{-1}((M-A)\times\{a\cdot b\}))$ of the group $\pi_1(R-(A\times[0, 1]))$ with another group. Since $\pi_1((M-A)\times\{1\})$ maps epimorphically to $\pi_1((M-A)\times[c, 1])$ it follows that $f((M-A)\times\{1\})$ maps epimorphically to $\pi_1(R-A\times[0, 1])$. Thus the hypotheses of (6.1) of [3] are satisfied and the homeomorphism h exists.

We extend to a homeomorphism of all of $M * v$ onto $f(M * v)$ by setting $h(m, t)=(t/b)\cdot h(m, b)$ for $0\leqq t\leqq b$. We show there is an isotopy f_t of f to h. Define

$$\begin{aligned} f_t(m, s) &= h(m, s), && t \leqq s \leqq 1 \\ &= h(t\cdot h^{-1}f(m, s/t)), && 0 \leqq s \leqq t,\ t > 0. \end{aligned}$$

Then for $m\in\partial M\cup A$, $f_t(m, s)=f(m, s)$; and $f_t(m, 1)=f(m, 1)$ for all m. Thus by an isotopy we replace f by h. Now h has the property that it maps $M\times\{b\}$ onto $M\times\{a\cdot b\}$. But R was triangulated so that $M\times\{a\cdot b\}$ in R got its induced structure from $M * v$. Thus h maps $M\times\{b\}$ piecewise linearly onto $M\times\{a\cdot b\}$. By the way h was extended to the rest of the cone, it is piecewise-linear from $M\times[0, b]$ onto $M\times[0, a\cdot b]$ as desired.

(4.8) THEOREM. *Let K and L be 3-complexes, and let $f\colon |K|\to|L|$ be a homeomorphism. Let $\varepsilon(x)$ be a positive continuous function on $|K|$, and let ρ be a metric on $|L|$. Suppose that f is piecewise-linear on a possibly empty subcomplex M of K. Then there exists an isotopy $f_t\colon |K|\to|L|$ so that*

(i) *$f_0=f$, f_1 is piecewise-linear.*

(ii) *$f_t \mid |M|=f \mid |M|$.*

(iii) *$\rho(f_t(x), f(x))<\varepsilon(x)$ for all $x\in|K|$.*

Proof. As in the proof of (4.6) we may assume that int $|M|=\varnothing$, and that every 2-simplex of M is in bK. If K_1 is the set of points of index less than 2, then by (4.4) we can $\varepsilon(x)$-isotope f to a homeomorphism which is piecewise-linear on K_1 without changing it on M. If A is the set of points of $|K|$ of local dimension less than 3, then $\bar{A}$ is the point set of a subcomplex of dimension at most 2. Notice that $\bar{A}-A\subset K_1$.

Keeping f unchanged on $|K|-A$ we can isotope it to a homeomorphism which is piecewise-linear on $\bar{A}$. If we replace M then by $M\cup(\bar{A}-A)$, $|K|$ by $|K|-A$, and $|L|-f(A)$ we may assume that K has constant local dimension 3, and that $f|K_1$ is piecewise-linear.

With these preliminaries out of the way let us describe the stages of the proof. We first use (4.6) to isotope $f|bK$ to a PL homeomorphism f_1 onto bL which preserves index and singularity with respect to K and L. Using (3.10) we lift the isotopy to $p^{-1}(bK)$, and then we use a collar neighborhood to extend to an isotopy of K which agrees with $\tilde{f}$ except on the collar. We find a further isotopy of $\tilde{f}$ to make it piecewise-linear in a neighborhood of $p^{-1}(bK)\cup M$. Punching out this neighborhood we are left with a manifold with boundary. Here theorems of Bing and Kister finish the isotopy to a piecewise-linear homeomorphism. Then (3.11) allows us to push everything back down to K and L, concluding the proof.

Step 1. Let $g_1=f|bK$. Let $K(2,s)$ be those points of index 2 and singularity s. Notice that g_1 Cl $[(K(2,s))]=$ Cl $[L(2,s)]$ and that Cl $[K(2,s)]-K(2,s)\subset K_1$ Using (4.6) we can $\varepsilon(x)$-isotope $g_1|$Cl $[K(2,s)]$ to a piecewise-linear homeomorphism g_0 of Cl $[K(2,s)]$ onto Cl $[L(2,s)]$ without changing $g_1|K_1\cup M$ in the process. Thus these isotopies fit together to give an isotopy g_t of g_1 to a piecewise-linear homeomorphism g_0 of $|bK|$ onto $|bL|$, which preserves index and singularity with respect to K and L.

Step 2. By (3.10) we can lift the isotopy g_t to an isotopy $\tilde{g}_t$ on $p^{-1}(|bK|)$. By (2.9) the complement in $\tilde{K}$ of $\tilde{K}_0$, the points of index 0, is a manifold with boundary. We let N be the $\tilde{K}$ closure of $p^{-1}(|bK\cup M|)\cap$ int $(\tilde{K}-\tilde{K}_0)$. This is a 1-complex which lies mostly in the interior of the manifold $\tilde{K}-\tilde{K}_0$ and on which we must not change $\tilde{f}$. We let P be the $\tilde{K}$ closure of the manifold boundary of $\tilde{K}-\tilde{K}_0$. We come now to the collar neighborhood. Consider the closed star of $P-N$ in the second barycentric subdivision $\tilde{K}''$ of $\tilde{K}$. It consists of a regular neighborhood [18] or [19] of the manifold boundary $P-N$, together with the points $P\cap N$.

The structure of St $(P-N;\tilde{K}'')$ around a point of $P\cap N$ is pictured in Figure 1. It is locally the cone over a finite family of annuli. On the other hand St $(P-N;\tilde{K}'')$ $-N$ is a collar neighborhood of $P-N$ in $|\tilde{K}|-(N\cup|\tilde{K}_0|)$. Choose a homeomorphism h of $(P-N)\times[0,1]$ onto the collar which respects the cone structure. That is, the level surfaces $h((P-N)=\{t\})$ should meet the cone in the cone over a circle less the vertex. We also require $h(x,0)=x$.

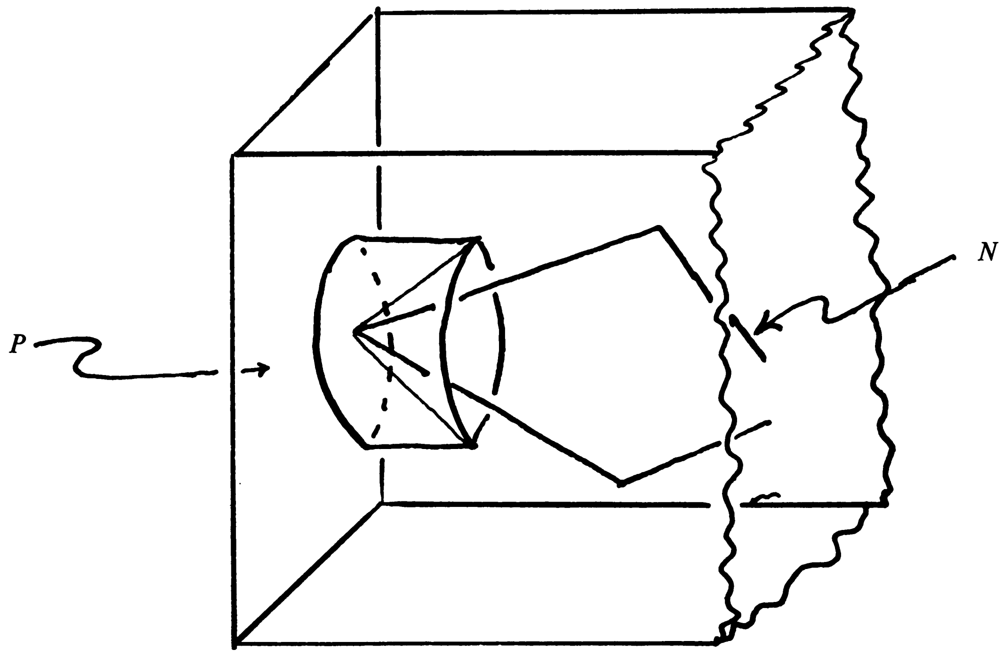

FIGURE 1

We extend $\tilde{g}_t$ to an isotopy of $\tilde{K}$ onto $\tilde{L}$ which we denote by $\tilde{f}_t$. Let,

$$\tilde{f}_t(h(x, s)) = \tilde{f}h(\tilde{f}^{-1}\tilde{g}_{t-s}(x), s), \qquad 0 \leqq s \leqq t,$$
$$\tilde{f}_t(p) = \tilde{f}(p) \quad \text{if } p \notin h(P - N \times [0, t]).$$

Since $\tilde{g}_0 = \tilde{f}|p^{-1}|bK|$ this is a well-defined isotopy. Since $h(x, 0) = x$, $\tilde{f}_t$ is an extension of $\tilde{g}_t$. Notice that $\tilde{f}_t$ agrees with $\tilde{f}$ on $p^{-1}|M|$.

Step 3. Using (4.7) we can, by a small isotopy, move $\tilde{f}_1$ to a homeomorphism $\tilde{f}_2$ which is PL in a neighborhood of the vertices of $p^{-1} \mid |M| \cup \tilde{K}_0$, without changing it on $p^{-1}(|bK \cup M|)$ in the process. Now punch out the neighborhood from $\tilde{K}$, and its $\tilde{f}_2$-image from $\tilde{L}$, getting manifolds with boundary $\tilde{K}^1$ and $\tilde{L}^1$. We continue to denote the restriction of $\tilde{f}_2$ to $\tilde{K}^1$, by $\tilde{f}_2$ and we notice that is PL on the boundary. With the exception of points in the boundary, $p^{-1}(|M \cup K_1|) \cap \tilde{K}^1$ is a family of properly embedded arcs on which $\tilde{f}_2$ is piecewise-linear (properly embedded means that their end points, and nothing else, lie in the boundary). The second regular neighborhood of such an arc is PL homeomorphic to the product of the arc with a disk, the end disks going into the boundary of the manifold. An argument analogous to (4.7) shows that we can isotope $\tilde{f}_2$ to a homeomorphism $\tilde{f}_3$ which is PL in a neighborhood of the arcs not moving points on the arcs or on the boundary in the process. Punch out this neighborhood and its $\tilde{f}_3$-image getting $\tilde{K}^2$ and $\tilde{L}^2$.

Step 4. We now have $\tilde{f}_3$ PL on the boundary of $\tilde{K}^2$ and this boundary contains $\tilde{K}^2 \cap p^{-1}(|bK \cup M|)$. By [1] we approximate $\tilde{f}_3$ arbitrarily closely by a PL homeomorphism $\tilde{f}_4$ which agrees with $\tilde{f}_3$ on the boundary of $\tilde{K}^2$. But Kister showed in [7] that two such homeomorphisms were isotopic by a small isotopy which agrees with them on the boundary. We extend this isotopy to $\tilde{K}^1$ by making it agree with $\tilde{f}_3$ where it was not already defined. In the same manner we extend all our isotopies to $\tilde{K}$.

Step 5. If $\tilde{f}_t$ is the extended isotopy from $\tilde{f}$ to $\tilde{f}_4$, then $\tilde{f}_t$ agrees with $\tilde{g}_t$ where both are defined and agrees with $\tilde{g}_1$ for $t > 1$ where both are defined. According to (3.11), $\tilde{f}_t$ covers an isotopy f_t of $|K|$ to $|L|$. Notice that f_4 is a PL homeomorphism and f_t agrees with f on $|M|$. Finally since $\tilde{f}_t$ could be taken arbitrarily close to $\tilde{f}$, we have that f_t is an $\varepsilon(x)$-approximation to f. Q.E.D.

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